

ON PERIODIC SOLUTIONS FOR A REDUCTION OF BENNEY CHAIN

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ABSTRACT. We study periodic solutions for a quasi-linear system, which is the so called dispersionless Lax reduction of the Benney moments chain. This question naturally arises in search of integrable Hamiltonian systems of the form $H = p^2/2 + u(q, t)$. Our main result classifies completely periodic solutions for 3 by 3 system. We prove that the only periodic solutions have the form of traveling waves, so in particular, the potential u is a function of a linear combination of t and q . This result implies that there are no nontrivial cases of existence of the fourth power integral of motion for H : if it exists, then it is equal necessarily to the square of the quadratic one. Our method uses two new general observations. The first is the genuine non-linearity of the maximal and minimal eigenvalues for the system. The second observation uses the compatibility conditions of Gibbons-Tsarev in order to give certain exactness for the system in Riemann invariants. This exactness opens a possibility to apply the Lax analysis of blow up of smooth solutions, which usually does not work for systems of higher order.

1. INTRODUCTION

Let $H = p^2/2 + u(q, t)$ be a Hamiltonian of a 1,5-degrees of freedom system with the potential which is assumed throughout this paper to be periodic function in both variables. There is a conjecture attributed to G.Birkhoff saying that the only integrable plane convex billiards are ellipses. The direct analog of this conjecture for the Hamiltonian system with 1,5 degrees of freedom would be the claim that the only integrable Hamiltonian functions of the form $H = p^2/2 + u(q, t)$ are those having the potential functions u which are periodic functions of the form $u = u(mq + nt)$. There are several attempts to approach this problem. Let me refer to the works [8],[2],[5],[13],[1] for various approaches to this circle of questions which remain outside the discussion of this paper.

In the present note we restrict this question to the search of the additional integrals which are polynomial with respect to the momenta variable p with the coefficients which are periodic functions of q and t . More precisely we want to find all those potential functions $u(q, t)$ for which there exists an additional function $F(p, q, t)$ invariant under the Hamiltonian flow (such an F is called the first integral of motion) which is a polynomial in the variable

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p of a given degree, say $n + 1$, having all the coefficients periodic in q and t . Write

$$F(p, q, t) = u_{-1}p^{n+1} + u_0p^n + u_1p^{n-1} + \cdots + u_n,$$

and substitute to the equation of conservation of F .

$$(1) \quad F_t + pF_q - u_qF_p = 0$$

Equating to zero the coefficients of various powers of p , one easily obtains the following information. The coefficient u_{-1} must be a constant, which will be normalized to be $\frac{1}{n+1}$. Also u_0 must be a constant, which we shall assume to be zero (this can be achieved by a linear change of coordinates on the configuration space \mathbb{T}^2). Moreover the coefficient u_1 satisfies $(u_1)_q = (u)_q$. Therefore, u_1 and u will be assumed to be equal (the addition of any function of t to the potential u does not change the Hamiltonian equations). Moreover, the column of the of the rest of the coefficients $U = (u_1, \dots, u_n)^t$ satisfy the following quasi-linear system of equations.

$$(2) \quad U_t + A(U)U_q = 0, \quad A(U) = - \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ (n-1)u_1 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2u_{n-2} & 0 & 0 & \cdots & 0 & -1 \\ u_{n-1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Notice that the derivative F_p of F coincides with the characteristic polynomial of $A(U)$.

In fact the system (2) is very well known among integrable systems community. This is the so called dispersionless Lax reduction of the moments Benney chain (see for example [6], [7], [12], [14] and references there in). There are many beautiful properties of this reduction. For example, this is a Hamiltonian system of Hydrodynamics type (in the sense of Dubrovin and Novikov see [9]. Moreover, it has infinitely many additional conservation laws¹. The most important property for this paper is that the system (2) is diagonalizable, i.e. can be written in the form Riemann invariants (see (3) below). Almost nothing is known, however, about the global existence of smooth solutions for this system. In the theory of quasi-linear hyperbolic PDEs it is a well known problem to prove the occurrence of blow up of smooth solutions. It was performed first by Lax [11] for 2 by 2 systems satisfying the so called genuine nonlinearity condition. His method relies heavily on the possibility to write the 2 by 2 system in the diagonal form, and also on the genuine non-linearity condition. Lax analysis was performed in [4] for 2 by 2 system of the form (2) where it was proved that the only periodic solutions for that case are constants. For systems of higher size the original method by Lax does not apply in general. We will show bellow two new observations concerning our quasi-linear system. The first is, that for the hyperbolic case, i.e. the case when all eigenvalues of the matrix $A(U)$ are distinct and real, it follows that the minimal and maximal eigenvalues of $A(U)$ are in fact genuinely non-linear in the sense of Lax (see Corollary 2.2

¹I was told by M.Pavlov that this is a well known fact among the specialists in integrable systems, see also [3] where it was rediscovered

bellow) . The second key fact is that the so called Gibbons-Tsarev compatibility system (the equation (d) of Lemma 2.1 and Corollary 2.3) provides certain "exactness" of the system (see the equation (6) and therefore enables one to perform the Lax original analysis for higher values of n .

Our main application of this approach in this paper is the following classification of smooth periodic solutions for the system.

Theorem 1.1. *Let $n = 3$. Then the only periodic solution of the quasi-linear system (2) are the traveling waves solutions, where u_1, u_2, u_3 do not depend on t .*

Corollary 1.2. *Let $F = 1/4p^4 + u_1p^2 + u_2p + u_3$ be a polynomial of degree four with periodic coefficients which satisfies the equation (1). Then there necessarily exists a quadratic integral, i.e. the energy H , and F is a function of H .*

In order to prove this theorem we shall divide between different regions: strictly Hyperbolic region Ω_h where all three eigenvalues of $A(U)$ are real and distinct, Elliptic region Ω_e , where two eigenvalues are complex conjugate and the third one is real, and finally the region of degeneracy Ω_0 , where at least two of the eigenvalues collide. It turns out that in all these regions the behavior of solutions can be understood completely. The proof of Theorem 1.1 is obtained just by patching together the information of these three cases. Organization of the paper is as follows. In the next Section we shall explain the two basic observations mentioned above concerning hyperbolic situation. In Sections 3, 4, 5 we study the regions $\Omega_h, \Omega_e, \Omega_0$ respectively for the case of $n = 3$. In Section 6 we prove formulas for derivatives of the eigenvalues and verify, for the sake of completeness, the Gibbons-Tsarev compatibility conditions.

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2. MAIN OBSERVATIONS

Let me denote by $\lambda_1, \dots, \lambda_n$ the roots of the polynomial F_p . And let $r_i = F(\lambda_i)$ be the corresponding critical values. The starting point for us is a beautiful classical theorem by MacLane, stating that the mapping $(u_1 \dots u_n) \mapsto (r_1, \dots, r_n)$ is in fact a global diffeomorphism between the domain of strict Hyperbolicity (that is the domain of all $U = (u_1, \dots, u_n)^t$ where all the roots of the polynomial F_p are real and distinct) with the domain of all possible critical values in \mathbb{R}^n that is of all those (r_1, \dots, r_n) such that the differences $(r_k - r_{k+1})$ have the sign $(-1)^{k+n}$ for all $k = 1, \dots, n-1$ (we refer to [10] for E.B. Vinberg's proof of this theorem and further results and discussions). According to this theorem (r_1, \dots, r_n) can be taken as regular global coordinates in this domain.

The importance of these coordinates for our system follow from the following computation. Substitute $p = \lambda_i$ into the equation (1). One gets the following diagonal system on the variables r_i (they are called Riemann invariants)

$$(3) \quad (r_i)_t + \lambda_i(r_1, \dots, r_n)(r_i)_q = 0, \quad i = 1, \dots, n$$

The derivatives of the roots λ_i with respect to the critical values r_i satisfy the following relations.

Lemma 2.1. *The following formulas hold true*

$$(a) \quad \partial_{r_i} \lambda_i = -\frac{1}{F_{pp}(\lambda_i)} \sum_{k=1, k \neq i}^n \frac{1}{\lambda_i - \lambda_k},$$

$$(b) \quad \partial_{r_i} \lambda_j = -\frac{1}{F_{pp}(\lambda_i)} \frac{1}{\lambda_j - \lambda_i}, \quad i \neq j,$$

$$(c) \quad F_{pp}(\lambda_i) u_{r_i} = 1,$$

$$(d) \quad u_{r_i r_k} = \frac{2u_{r_i} u_{r_k}}{(\lambda_i - \lambda_k)^2}.$$

Remark 1. The condition (d) is in fact the so called Gibbons Tsarev compatibility condition (see [7])(note there is a missprint in their formula-the factor 2 is missing). I didn't find however the formula (a) in any paper on the subject. We suggest the proof of all of them in Section 6 in a very short way.

Corollary 2.2. *In the strictly hyperbolic region $\lambda_1 < \lambda_2 < \dots < \lambda_n$ the maximal and minimal eigenvalues are genuinely nonlinear in the sense of Lax:*

$$\partial_{r_1} \lambda_1 \neq 0, \partial_{r_n} \lambda_n \neq 0$$

Another important consequence of the Gibbons-Tsarev conditions is the following

Corollary 2.3. *In the Hyperbolic region introduce the functions*

$$G_i = -\frac{1}{2} \log |u_{r_i}| = \frac{1}{2} \log |F_{pp}(\lambda_i)|$$

Then it follows from the lemma that

$$\partial_{r_j} G_i = -\frac{u_{r_i r_j}}{2u_{r_i}} = -\frac{u_{r_j}}{(\lambda_i - \lambda_j)^2} = \frac{(\lambda_i)_{r_j}}{\lambda_i - \lambda_j}$$

In order to perform the blow up analysis for our system we shall differentiate the quantities $w_i, w_i = (r_i)_q$ along the integral curves of the family λ_i . They are, by definition, the integral curves of the equation $\dot{q} + \lambda_i(q, t) = 0$ on \mathbb{T}^2 . Let $v_i = (1, \lambda_i(q, t))$ be the i -th vector field and let $L_{v_i} = \partial_t + \lambda_i \partial_q$ denotes the Lie derivative along the field v_i . Differentiating with respect to q the i -th equation of (3) one gets the following

$$(4) \quad L_{v_i}(w_i) + w_i^2 (\lambda_i)_{r_i} + w_i \sum_{j \neq i} (\lambda_i)_{r_j} (r_j)_q = 0$$

Notice that by definition:

$$L_{v_i} r_j = (r_j)_t + \lambda_i (r_j)_q$$

Subtract from this expression the j -th equation of (3)

$$0 = (r_j)_t + \lambda_j(r_j)_q$$

one verifies that

$$(r_j)_q = \frac{L_{v_i} r_j}{\lambda_i - \lambda_j}.$$

Substitution of this expression into (4) leads to:

$$(5) \quad L_{v_i}(w_i) + w_i^2(\lambda_i)_{r_i} + w_i \sum_{j \neq i} (\lambda_i)_{r_j} \frac{L_{v_i} r_j}{\lambda_i - \lambda_j} = 0$$

Therefore it follows from the last (Corollary 2.3) that (5) can be rewritten in the following way

$$L_{v_i}(w_i) + w_i^2(\lambda_i)_{r_i} + w_i \sum_{j \neq i} (G_i)_{r_j} L_{v_i} r_j = 0.$$

Therefore, taking into account the i -th equation of (3) we get:

$$(6) \quad L_{v_i}(w_i) + w_i^2(\lambda_i)_{r_i} + w_i L_{v_i} G_i = 0$$

Multiplying by $\exp G_i$ this equation one rids of the linear term as follows:

$$(7) \quad L_{v_i}((\exp G_i)(w_i)) + (\exp(-G_i)(\lambda_i)_{r_i})(\exp(2G_i)w_i^2) = 0$$

Using the explicit expression for G_i of Corollary 2.3 and denoting

$$z_i = |F_{pp}(\lambda_i)|^{1/2} w_i = |F_{pp}(\lambda_i)|^{1/2} (r_i)_q, \quad K_i = |F_{pp}(\lambda_i)|^{-1/2} (\lambda_i)_{r_i}$$

we get the following equation, which is crucial for the analysis of the blow up of the solution.

$$(8) \quad L_{v_i} z_i + K_i z_i^2 = 0, \quad i = 1, \dots, n$$

As an immediate consequence of this equation we state the following

Theorem 2.4. *Let $U = (u_1, \dots, u_n)^t$ be a periodic solution of the quasi-linear system (2) corresponding to the strictly Hyperbolic regime, i.e. all eigenvalues are real and distinct: $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Then the Riemann invariants r_1 and r_n corresponding to the minimal and maximal eigenvalues are constants.*

Proof. This fact follows immediately from the equation (8). Indeed, in the region of strict hyperbolicity the $F_{pp}(\lambda_i)$ does not vanish and so, by the genuine non-linearity of λ_1 and λ_n the functions K_1, K_n are bounded away from zero. Then it follows from the explicit formula for the solutions of (8) that the only solution which does not explode in a finite time is $z_1, z_n = 0$. Thus $(r_1)_q = (r_n)_q = 0$ and the equations (3) imply that r_1 and r_n must be constants. This yields the result. \square

A refinement of this argument is the content of the next section on the Hyperbolic region, for 3 by 3 system. In what follows we shall assume that $n = 3$, i.e. the system is 3 by 3. Let me denote by Ω_h be the region of strict Hyperbolicity and Ω_e be the region, where there are two complex conjugate eigenvalues for $A(U)$. The complement, $\mathbb{T}^2 - (\Omega_h \cup \Omega_e)$ is the set of those points (q, t) where the matrix $A(U)$ has at least two equal eigenvalues. We shall denote this set Ω_0 . And finally, Ω_{00} will denote the set of maximal

degeneration, i.e. where all three eigenvalues are equal and thus equal to zero (since the sum of all the three eigenvalues vanishes).

3. HYPERBOLIC REGION Ω_h .

Before stating the main result of this section, let me rewrite the formulas of the Lemma 2.1 for the case of 3 by 3 system.

In this case since $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we get the following simplifications

$$(\lambda_i)_{r_i} = -\frac{3\lambda_i}{F_{pp}(\lambda_i)^2}, \quad K_i = -\frac{3\lambda_i}{|F_{pp}(\lambda_i)|^{5/2}}, \quad F_{pp}(\lambda_i) = \prod_{j \neq i} (\lambda_i - \lambda_j)$$

For the case $n = 3$ we have the following refinement of the theorem of the previous section.

Theorem 3.1. *Let $U = (u_1, u_2, u_3)^t$ be a periodic solution of the system (2), and let $\Omega_h \subseteq \mathbb{T}^2$ be the domain of strict Hyperbolicity. Then the Riemann invariants r_1, r_3 are constants in every connected component of the Hyperbolic domain Ω_h .*

Proof. We give the proof for r_1 , the other case is analogous. The first step of the proof is the fact that the derivatives $(r_i)_q$ are bounded functions on the domain Ω_h . Indeed, by definition,

$$r_i = F(\lambda_i, q, t) \Rightarrow (r_i)_q = F_q(\lambda_i, q, t)$$

and thus by the periodicity of the coefficients of the polynomial F , all roots λ_i of the derivative F_p are bounded and so are the derivatives $(r_i)_q$. Consider the integral curves of the λ_1 -family in the domain Ω_h . Suppose that such a curve approaches the boundary of Ω_h , then $F_{pp} \rightarrow 0$ while $(r_i)_q$ stays bounded. Therefore, it follows that

$$z_i = (r_i)_q |F_{pp}(\lambda_i)|^{1/2} \rightarrow 0.$$

I claim that then z_1 equals zero identically in Ω_h . If, for example, z_1 is positive at a point (q_0, t_0) then by the equation (8) and the fact that K_1 is positive, we have that z_1 is a decreasing function of time and thus in the backward time along the integral curve cannot approach the boundary, because on the boundary z_1 vanishes. On the other hand, if the integral curve stays inside Ω_h forever in the backward time, then the function K_1 stays bounded away from zero, and therefore the blow up of the solution must occur in a finite (backward) time. Thus z_1 can not be positive. The opposite case, when $z_1(q_0, t_0)$ can not be negative is completely analogous.

This argument in a more precise form looks as follows. Denote by M_1, M_2 positive constants such that

$$|(r_1)_q| < M_1, \quad \sqrt{|F_{pp}(\lambda_i)|} < M_2.$$

Then for any backward time along the integral curve the monotonicity of z_1 implies that

$$M_1 \sqrt{|F_{pp}(\lambda_1)|} > (r_1)_q \sqrt{|F_{pp}(\lambda_1)|} = z_1 \geq z_1(q_0, t_0)$$

and therefore

$$\sqrt{|F_{pp}(\lambda_1)|} > \frac{z_1(q_0, t_0)}{M_1}.$$

In addition, λ_1 can not be too close to zero. Indeed, if $0 \leq -\lambda_1 < a$, then by the zero sum condition also $|\lambda_2| < a$ and $0 \leq \lambda_3 < 2a$. Then one would get

$$\sqrt{|F_{pp}(\lambda_i)|} = ((\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1))^{1/2} < \sqrt{6}a.$$

So together with the previous estimate this implies that $|\lambda_1| \geq \frac{z_1(q_0, t_0)}{\sqrt{6}M_1}$. Then

$$K_1 = \frac{3|\lambda_1|}{((\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1))^{5/2}} \geq \frac{3z_1(q_0, t_0)}{\sqrt{6}M_1M_2^5} > 0$$

So K_1 is bounded away from zero, and again by the explicit formula for the solution of (8) it explodes in a finite backward time. This proves the claim that z_1 vanishes identically in Ω_h . Thus $z_1, (r_1)_q \equiv 0$, and so by the equations (3) r_1 must be constants. \square

The next theorem describes completely the solutions of the system in the Hyperbolic region Ω_h .

Theorem 3.2. *Either the solution $U = (u_1, u_2, u_3)^t$ is a constant solution for (2) on \mathbb{T}^2 , or the region Ω_h is a union of strips on the torus parallel to the t -axes and the following relations hold*

$$u_1 = u_1(x), u_2 \equiv 0, u_3 = u_1^2 + \text{const},$$

so that the polynomial F equals (up to a constant) in Ω_h to the square of the Hamiltonian

$$F = \left(\frac{p^2}{2} + u_1(q) \right)^2 + \text{const}.$$

Proof. Let me note, that since r_i are successive critical values of the polynomial F_p , then everywhere in Ω_h holds $r_2 > r_1, r_2 > r_3$. On the boundary $\partial\Omega_h$ two of the eigenvalues collide, say λ_2 collides with λ_1 (or with λ_3 , or maybe both), and hence $r_2 = r_1$ (or $r_2 = r_3$). By the previous theorem r_1, r_3 are constants in Ω_h , and in addition r_2 has constant values along the integral curves of the λ_2 -family. Then it follows, that non of these curves can approach the boundary, because otherwise this would give $r_1 = r_2$ in the inner points. Moreover, the function r_2 satisfies the equation

$$(r_2)_t + \lambda_2(r_2)(r_2)_q = 0,$$

where $\lambda_2(r_2)$ depends only on r_2 , since r_1, r_3 are constants in Ω_h . Therefore, the characteristics of this equation are the straight lines in the (q, t) -plane, and thus there are two possible cases. The first case is that there exist two intersecting straight lines of the family. Then r_2 has to be constant everywhere (since r_2 has constant values along characteristics, and any straight line intersects at least one of the two intersecting characteristics). So in this case all r_1, r_2, r_3 are constants everywhere in Ω_h , and hence also the coefficients of the polynomial u_1, u_2, u_3 are constants. In such a case Ω_h must be the whole \mathbb{T}^2 .

In the second case all the characteristic straight lines are parallel with the same $\lambda_2 = \mu = \text{const}$, and thus the domain Ω_h in this case is a union

of parallel strips with the slope μ . Next we claim that if the solution $U = (u_1, u_2, u_3)^t$ is not constant in Ω_h , then $\mu = 0$. To see this, let me recall that

$$F_p(\mu) = \mu^3 + 2u_1\mu + u_2 = 0,$$

and therefore

$$(9) \quad u_2 = -2\mu u_1 - \mu^3$$

Substituting (9) into the first equation of the system (2) we have

$$(u_1)_t = -(u_2)_q = 2\mu(u_1)_q$$

In addition we have that along any characteristic straight line of the family λ_2 the values of r_1, r_2, r_3 are constants, and then also $\lambda_1, \lambda_2, \lambda_3, u_1, u_2, u_3$, because r_1, r_2, r_3 are genuine coordinates. In order to prove the claim let us assume that on the contrary $\mu \neq 0$. Then u_1 has to be globally constant in Ω_h because it satisfies the following two equations

$$(10) \quad (u_1)_t - 2\mu(u_1)_q = 0, \quad (u_1)_t + \mu(u_1)_q = 0$$

But then, by (9), also u_2 is constant in Ω_h . Therefore $\lambda_1, \lambda_2, \lambda_3$ are all constants, since the polynomial F_p has constant coefficients. Also since $u_3 = r_1 - F(\lambda_1)$ then also u_3 must be a constant. So the solution is in fact a constant solution contradicting the assumption of the claim.

Thus, we get that $\mu = 0$. In this case all characteristics of the second family are parallel to the t -axes and the region Ω_h is the union of strips parallel to the t -axes. Moreover by (10) u_1, u_3 have to be the functions on q only and by (9) $u_2 \equiv 0$.

Therefore $F_p = p(p^2 + 2u_1)$, and so

$$F = \frac{1}{4}p^4 + u_1p^2 + u_3 = \left(\frac{p^2}{2} + u_1\right)^2 + u_3 - u_1^2,$$

and since r_1, r_3 are constants, then $u_3 - u_1^2 = \text{constant}$, and we are done. Notice that on the boundary of Ω_h , u_1 vanishes and so $\partial\Omega_h \subseteq \Omega_{00}$. \square

4. ELLIPTIC REGION Ω_e

Consider now the region Ω_e where the polynomial F_p has two complex conjugate roots, say $\lambda_{1,2} = \alpha \pm i\beta$ with $\beta > 0$ in Ω_e , and $\lambda_3 \in \mathbb{R}$. In this case $r_{1,2}$ are also complex conjugate, say $r_{1,2} = v \pm iw$ and r_3 is real. Notice, that for the points of the boundary of Ω_e we have $\lambda_1 = \lambda_2 = -\lambda_3/2$ are real, $r_1 = r_2$ are real also, and so $\beta = 0, w = 0$. For the region Ω_e we have the same description of solutions as in the strictly Hyperbolic domain, but for completely different reasons.

Theorem 4.1. *Either $U = (u_1, u_2, u_3)^t$ is a constant solution for the system (2) on the whole \mathbb{T}^2 , or the region Ω_e is a union of strips parallel to the t -axes on the torus and the following relations hold*

$$u_1 = u_1(q), u_2 \equiv 0, u_3 = u_1^2 + \text{const}$$

So that the polynomial F equals (up to a constant) in Ω_e to the square of the Hamiltonian

$$F = \left(\frac{p^2}{2} + u_1(q)\right)^2 + \text{const}.$$

Proof. The Riemann invariants $r_{1,2} = v \pm iw$ satisfy the equations (3), which are equivalent to the following elliptic system on their real and imaginary parts.

$$\begin{aligned} v_t + \alpha v_q - \beta w_q &= 0 \\ w_t + \beta v_q + \alpha w_q &= 0. \end{aligned}$$

This is an elliptic system, and therefore, since w vanishes on the boundary, then by the strong maximum principle the function w must vanish identically in the whole Ω_e . Substituting back to the elliptic system we get that v is a constant everywhere Ω_e . Therefore, $\lambda_{1,2}$ are roots of the polynomials $F - v$ and of F_p . Then,

$$F = \frac{1}{4}(p - \lambda_1)^2(p - \lambda_2)^2 + v = \frac{1}{4}((p - \alpha)^2 + \beta^2)^2 + v.$$

Moreover, α vanishes because F does not contain the cubic terms. Therefore

$$F = (p^2/2 + \beta^2/2)^2 + v, \quad \lambda_{1,2} = \pm i\beta, \lambda_3 = 0$$

This situation can be completely analyzed, because in this case the quadratic function $\tilde{F} = p^2/2 + \beta^2/2$ is the conserved quantity. But then, by the equation (1) for \tilde{F} we have:

$$\begin{aligned} (\beta^2)_t &= 0 \\ (\beta^2/2 - u_1)_q &= 0. \end{aligned}$$

Notice, that on the boundary of Ω_e we have $\beta = u_1 = u_2 \equiv 0$. By the first equation β vanishes on any line $q = \text{const}$ whenever it crosses the boundary, and it is a constant on any line which lies entirely inside Ω_e . This yields immediately that Ω_e is in fact union of strips parallel to the t axes and $\beta = \beta(q)$ and also $\beta^2/2 - u_1 \equiv 0$ because on the boundary both β and u_1 vanish. So we proved

$$F = (p^2/2 + u_1(q))^2 + \text{const}$$

and this completes the proof \square

5. DEGENERATE REGIONS Ω_0, Ω_{00}

It follows from the description of Ω_e and Ω_h that the degenerate region Ω_0 is a union of strips parallel to the t axes and moreover every point of the boundary of each strip belongs, in fact to Ω_{00} . Then, it follows that $\Omega_0 - \Omega_{00}$ is an open set. We claim next that the degeneration is maximal everywhere, i.e. $\Omega_0 = \Omega_{00}$. Indeed, consider an integral curve of the λ_3 -family lying inside this open set $\Omega_0 - \Omega_{00}$. Then r_3 has constant values along the curve, and therefore it can not approach the boundary of the region $\Omega_0 - \Omega_{00}$, since otherwise inside one would get $r_1 = r_2 = r_3$ inside the region, which contradicts the assumptions. Thus the whole characteristic stays inside the region $\Omega_0 - \Omega_{00}$. But then, exactly as above, in the proof of the Theorem 3.1, we have that the derivative $(r_3)_q$ must explode in a finite time unless it vanishes. And therefore, in the whole region $\Omega_0 - \Omega_{00}$, r_3 is a constant and therefore $\Omega_0 = \Omega_{00}$, and we are done.

6. PROOF OF THE LEMMA 2.1

It is rather simple to prove (a) and (b) of the lemma. By the definitions of λ_i and r_i we have

$$F_p(\lambda_i) = 0, \quad F(\lambda_i) = r_i$$

Differentiate these two formulas with respect to r_j . Notice that the roots λ_i depend on r_j and also the coefficients u_i of the polynomial F . We get

$$(11) \quad F_{pp}(\lambda_i)(\lambda_i)_{r_j} + F_{pr_j}(\lambda_i) = 0, \quad F_{r_j}(\lambda_i) = \delta_{ij}$$

Then write the following identity

$$(12) \quad F_{r_i}(p)(p - \lambda_i) = (u_1)_{r_i} F_p(p)$$

which becomes clear, if one notice that on both sides there are polynomials of the same degree n with the same leading coefficient $(u_1)_{r_i}$, and both having $\lambda_1, \dots, \lambda_n$ as the roots (the right hand side-just by definition, and the left hand side-by (11)).

Differentiate this identity with respect to p to obtain

$$(p - \lambda_i)F_{pr_i} + F_{r_i} = (u_1)_{r_i} F_{pp}$$

Substitute in this formula $p = \lambda_i$ one arrives to (c) of the lemma immediately. Substitute $p = \lambda_j$ and take into account (11), then one proves (b).

In order to prove (a) let me use the relation $F_{r_j}(\lambda_i) = \delta_{ij}$ of (11) in order to conclude, that F_{r_i} is, in fact, identical to the i -th Lagrange interpolation polynomial:

$$F_{r_i} = \prod_{s \neq i} \frac{(p - \lambda_s)}{(\lambda_i - \lambda_s)} = l_i(p)$$

Differentiate this identity with respect to p at the point λ_i :

$$F_{pr_i}(\lambda_i) = (l_i)_p(\lambda_i) = \sum_{s \neq i} \frac{1}{(\lambda_i - \lambda_s)}$$

Using (11) we obtain

$$(\lambda_i)_{r_i} = -\frac{F_{pr_i}(\lambda_i)}{F_{pp}} = -\frac{1}{F_{pp}(\lambda_i)} \sum_{s \neq i} \frac{1}{(\lambda_i - \lambda_s)}$$

This gives the proof of (a)

In order to derive (d) write (b) in the form

$$(\lambda_j)_{r_i} = u_{r_i}/(\lambda_i - \lambda_j)$$

Following Gibbons, Tsarev ([7]) differentiate this formula with respect to r_k to get

$$(13) \quad (\lambda_j)_{r_i r_k} = \frac{u_{r_i r_k}}{(\lambda_i - \lambda_j)} - \frac{u_{r_i} u_{r_k}}{(\lambda_i - \lambda_j)(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}$$

Now change the order of the indices i and k in (13) to have

$$(14) \quad (\lambda_j)_{r_k r_i} = \frac{u_{r_k r_i}}{(\lambda_k - \lambda_j)} - \frac{u_{r_k} u_{r_i}}{(\lambda_k - \lambda_j)(\lambda_i - \lambda_k)(\lambda_i - \lambda_j)}$$

Subtract now (13) and (14). One gets (d) of the lemma.

7. CONCLUDING REMARKS AND QUESTIONS

1. It would be very interesting to generalize the analysis presented here for 3 by 3 system to the case of higher orders. In general, there are much more cases of degenerations of eigenvalues for higher n . This makes the analysis much harder. However, the case of strictly Hyperbolic solutions seems to be tractable.

2. Besides the question on periodic solutions for the quasi-linear system, there is interesting question to understand the global existence of smooth (not necessarily periodic in time) solutions, having Hyperbolic initial data. In other words assume, that we are given for $t = 0$ the initial condition, say periodic functions in q $U_0(q) = (u_{10}(q), \dots, u_{n0}(q))$, such that the eigenvalues of the matrix $A(U_0)$ are real and distinct. Now switch on the dynamics. The question is if there are solutions existing for all times, or blow up of the solutions can be established.

3. One of the key ingredients of the proof in this paper was the observation that the (semi-)Hamiltonian property of the quasi-linear system allows one to push forward the Lax analysis of the blow up of solutions. It would be interesting to know, what can be said in this perspective for other reductions of the Benney chain.

4. It was proved by Tsarev ([14]), that in principle, the solutions for the system can be obtained by the so called generalized hodograph method. This method however, is very implicit and therefore, it is not clear to me how it can be used in order to answer the question of long time existence.

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